Quantum-state representation based on actual measurements — Beyond the Bloch vector for N-level systems —

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本稿において測定量に基づく量子状態表現のクラスを与える：すなわち，(A1) 双対基底の期待値による状態表現，(A2) 規格化条件を取り込んだ表現，(A3)Bloch ベクトル，であり，それらは (A1) ⊃ (A2) ⊃ (A3) という包含関係で結ばれる．特に (A2) の段階に着目し，その状態空間が基底の力学量の最小固有値により決定されることを示す．応用の一例として，Bloch ベクトルの状態空間が持つ双対構造を紹介する．

I. INTRODUCTION

A state is one of the most important notions in physics in order to describe physical phenomena with basic theory. In order to follow the dynamical aspects based on the unified equation, the changes of the phenomena can occur depending on the changes of the various states even within the same system. This is no exception in Quantum theory where many basic notions had to be changed from classical ones. Quantum state is usually represented with a density operator ρ — a positive operator with unit trace on the associated Hilbert space $\mathcal{H}$. However, different from classical representation of state, the quantity itself is an abstract operator which does not have a direct connection between experimental data; in stead of this, the appropriate algorithm to derive expectation values from a given density operator ρ are given: the expectation values $\langle A\rangle$ of observable A, which is represented by a self-adjoint operator on $\mathcal{H}$, can be calculated by the formula $\langle A\rangle = \operatorname{tr}\rho A$ where tr is a trace operation with complete orthonormal systems in $\mathcal{H}$. This abstraction of state has an advantage to include all the information of state and to treat all the observables equally (unlike classical representation). On the other hand, the following problem inevitably happens: If one does not have any information about the quantum state under consideration and wants to determine it with some experiments, how one can do it; what kinds of observables should be observed and how to reconstruct the density operator by their experimental data? All these are non-trivial problems even in principle and sometimes called Pauli problem. From this point of view, it is valuable to consider quantum-state representation based on actual measurements, expectation values in quantum cases, since then the Pauli problem does not exist from the beginning. In particular, if one knows the structure of the state space, it enables us to capture comprehensive and global aspects, in connection with experiments, of not only the state but also the dynamics. The Bloch-vector representation in 2-level systems (spin system) is one of the cases in point: It is a 3-dimensional real vector whose components are expectation values of spins x, y and z, and the state-space is a unit ball in $\mathbb{R}^3$ known as the Bloch ball [1]. Any dynamics such as Larmor precession movement and the decoherence process can be easily imagined in the ball and all the physical information of the state are captured intuitively thorough the simple geometry of the space. It is known that this notion can be easily generalized to arbitrary N-level systems [2] but it has become clear that the geometry of the state spaces is enormously complex [3]. Due to this complexity, it is thought that the Bloch-vectors for N-levels are not as useful as in 2-levels. Our goal in this letter is twofold: firstly by imposing natural conditions, we classify the state representation based on actual measurements. Since there is no uniqueness in choosing observables to be measured, this classification clarifies the reason of the choice. Hence our classification has a hierarchy $(A1) \supset (A2) \supset (A3)$ where $(A3)$ gives the Bloch-vector representation, this also has a possibility to find a new representation which has a simple geometry in its state space. Secondly, we find a dual structure in the complex space of the Bloch vector, which we believe useful to capture its geometrical aspect. In the following, we restrict ourselves in N-level systems for simplicity, but we notice that the class (A1) includes infinite dimensional cases unlike the Bloch vector representation.

II. CLASS OF THE QUANTUM-STATE REPRESENTATION

(A1) Representation based on dual base

The set of self-adjoint operator on $\mathcal{H}$ forms a real Hilbert space in terms of the Hilbert-Schmidt inner product: $(A, B) \equiv \operatorname{tr} AB$ with dimension N^2. Let $A_i = A_i^\dagger$ $(i = 1, \dots, N^2)$ be one of its base, then there exists dual base $\tilde{A}_i = \tilde{A}_i^\dagger$ $(i = 1, \dots, N^2)$ uniquely such that $(\tilde{A}_i, A_j) = N\delta_{ij}$. Since a density operator ρ is also self-adjoint, it can be expanded with the base and one obtains

$$\rho = \frac{1}{N}\sum_{i=1}^{N^2} (\tilde{A}_i, \rho) A_i = \frac{1}{N}\sum_{i=1}^{N^2} \langle \tilde{A}_i\rangle A_i, \tag{1}$$

where use has been made that $(\tilde{A}_i, \rho) = \operatorname{tr}\tilde{A}_i\rho = \langle\tilde{A}_i\rangle$.

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From this fact, one can consider $\boldsymbol{a} \equiv (\langle \tilde{A}_1 \rangle, \dots, \langle \tilde{A}_{N^2} \rangle)$, which is actual measurement in principle, a quantum state where the corresponding density operator is given by Eq. (1). This is our basic idea; that is to expand density operator by some observables with its coefficients being expectation values. Concerning the state space, one has to consider not only positive but also unit-trace condition for the operator of the form Eq. (1). However, the unit-trace condition is a constraint which decreases the degrees of freedom and can be naturally included in the following manner.

(A2) **Representation with normalization**

As a base in the class (A1), let us chose $\mathbb{1} = A_{N^2}$ (unit operator) and others $A_i (i = 1, \dots, N^2-1)$ being orthogonal to $\mathbb{1}$: namely, (a) $A_i^\dagger = A_i$ and (b) $(\mathbb{1}, A_i) = \mathrm{tr} A_i = 0$. Mathematically, they are generators of $SU(N)$. In this case, $\tilde{A}_{N^2} = A_{N^2}$ and the expansion Eq. (1) becomes

$$\rho = \frac{1}{N}(\mathbb{1} + \sum_{i=1}^{N^2-1} \langle \tilde{A}_i \rangle A_i), \tag{2}$$

where unit trace condition $a_0 = \mathrm{tr}\,\rho = 1$ is naturally included and $\boldsymbol{a} \equiv (\langle \tilde{A}_1 \rangle, \dots, \langle \tilde{A}_{N^2-1} \rangle)$ represents a quantum-state. Hence it is easier to investigate the state space than the general class (A1); we only have to consider the positivity condition of the operator Eq. (2).

(A3) **Bloch vector representation with N-levels**

By imposing also the orthogonality condition in class (A2) between A_i $(i = 1, \dots, N^2 - 1)$s, i.e., $\mathrm{tr}\, A_i A_j = N\delta_{ij}$, hence $\tilde{A}_i = A_i$, this gives the final class (A3) and this is nothing but the Bloch-vector representation for N-level systems, which is a generalization of the famous Bloch-vector for 2-level systems.

By their constructions, the class has the following inclusive relation: (A1) $\supset$ (A2) $\supset$ (A3). In the next section, we provide a characterization of the state space of the class (A2), hence also (A3).

III. STATE SPACE OF (A2) AND (A3)

Theorem 1 [4] The state space of class (A2) is given by

$$\mathcal{S}(\{A_i\}) = \{\boldsymbol{a} = r\boldsymbol{n} \in \mathbb{R}^{N^2-1} : r \leq \frac{1}{|m(A_{\boldsymbol{n}})|}\}, \tag{3}$$

where $\boldsymbol{n}$ is a direction vector and $m(A_{\boldsymbol{n}})$ is the minimum eigenvalue of $A_{\boldsymbol{n}} \equiv \sum_{i=1}^{N^2-1} n_i A_i$.

[**Sketch of the proof**] Since the unit-trace condition has already satisfied in class (A2), we only have to check the positivity condition of the operator $\rho = \frac{1}{N}(\mathbf{1}_N + a_i A_i) = \frac{1}{N}(\mathbb{1}_N + rA_{\boldsymbol{n}})$, where $a_i = rn_i$ $(r^2 \equiv \sum a_i^2, n_i \equiv a_i/r)$. This is equivalent to $\inf\langle\phi|\frac{1}{N}(\mathbb{1} + rA_{\boldsymbol{n}})|\phi\rangle \geq 0$ where infimum is taken over all the unit vectors $|\phi\rangle$ of $\mathcal{H}$. From $m(A_{\boldsymbol{n}}) = \inf\langle\phi|A_{\boldsymbol{n}}|\phi\rangle < 0$ due to $A_i^\dagger = A_i$ and $\mathrm{tr}\, A_{\boldsymbol{n}} = 0$, $A_{\boldsymbol{n}} \neq 0$, the condition becomes $r \leq \frac{1}{|m(A_{\boldsymbol{n}})|}$ ■

Theorem 1 tells us that all the information of the state-space for (A2) and (A3) is included in minimum eigenvalue of $A_{\boldsymbol{n}}$ in each direction $\boldsymbol{n}$. From this fact, if one could find a base for (A2), hence generator of $SU(N)$, which has minimum eigenvalues with simple condition, it gives more useful state-representation in the sense that the state space has a simple geometry than the Bloch vector representation.

IV. DUAL STRUCTURE OF THE BLOCH-VECTOR SPACE

Next, we apply theorem 1 especially to the Bloch-vector (A3) and find a dual structure in the complex geometry. Before that, we recall some of the known character of the set of the Bloch vector $B_N(\mathbb{R}^{N^2-1})$; it is closed convex set including the origin of $\mathbb{R}^{N^2-1}$ as a maximum mixed state; the maximum radius of the ball which is included in $B_N(\mathbb{R}^{N^2-1})$ is $r_l = \sqrt{N-1}$; while the minimum radius of the ball which includes $B_N(\mathbb{R}^{N^2-1})$ is $r_s = 1/\sqrt{N-1}$ where the state on the ball is pure state: $D_{r_s}(\mathbb{R}^{N^2-1}) \subseteq B_N(\mathbb{R}^{N^2-1}) \subseteq D_{r_l}(\mathbb{R}^{N^2-1})$, where we denote $D_r(\mathbb{R}^{N^2-1})$ a ball with radius r with its center being origin. Only when $N = 2$, $r_l = r_s$ and $B_2(\mathbb{R}^3)$ becomes a ball (Bloch ball).

Theorem 2 [4] The Bloch-vector space has a dual property: If the space is sticking out to the large ball D_{r_l} in some direction, the space can only reach to small ball D_{r_s} in the opposite direction; and vice versa. (We refer [4] for the proof).

This gives us overall picture of the Bloch-vector space. Physical meaning of this theorem is interesting; since the radius r of the Bloch vector is related to the purity $P(\rho) \equiv \mathrm{tr}\rho^2$ as $P(\rho) = \frac{(1+r^2)}{N}$, if there is a pure state in one direction, in opposite direction one has to give up the high purity.

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